Wave diagrams for ideal 2-fluid plasmas

Rony Keppens

Centre for mathematical Plasma Astrophysics
KU Leuven
MHD wave signals

- **static homogeneous plasma**: slow, Alfvén, fast wave pairs
  - the **phase speed diagrams** quantify for every angle $\theta$ between $\mathbf{k}$ and $\mathbf{B}$ how far a plane wave can travel in fixed time
  - point perturbation leads to the related **group diagram**, found from a Huygens construction on the phase speed diagram (constructive interference of all plane waves)
Phase and group diagrams [G&P, CUP, 2004]

Friedrichs diagrams (schematic) parameter $c_s/b = \frac{1}{2} \gamma \beta$, $\beta \equiv 2p/B^2$

Phase diagram (plane waves) Group diagram (point disturbances)
• locally perturb homogeneous magnetized plasma at rest
  \[ \Rightarrow \gamma = \frac{5}{3}, \rho = 1, \rho_{\text{th}} = 0.6 \text{ and } B = 0.9\hat{e}_x \ (c_s = 1, b = 0.9) \]
  \[ \Rightarrow (x, y) \in [-0.5, 0.5]^2 \text{ in 2.5D ideal MHD, include } v_z, B_z \]
  \[ \Rightarrow \text{perturb at origin with } \delta \rho = 0.1, \delta v_z = 0.01 \text{ and } \delta \rho_{\text{th}} = 0.06 \]
• MHD counterpart of \textit{‘throwing a stone in a puddle’}

\[ \Rightarrow \text{entropy, total pressure, } B_z \text{ at finite time} \]
Extension to Hall-MHD

- Hall-MHD: ion dynamics (massless $e$) in charge-neutral plasma $n_e = Zn_i$, where speeds $\mathbf{v} = \mathbf{u}_i$ and $\mathbf{u}_e = \mathbf{v} - (en_e)^{-1}\mathbf{j}$ and $\rho = n_i m_i$
  
  \[ \Rightarrow \] induction equation modifies to:

\[
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times [(\mathbf{v} - \frac{m_i}{Ze\rho}\mathbf{j}) \times \mathbf{B}] = 0
\]

\[ \Rightarrow \] introduces ion inertial length $\delta_i \equiv c/\omega_{pi}$, obtain DR

\[
(\omega^2 - k^2b^2) \left[ \omega^4 - k^2(b^2 + c_s^2)\omega^2 + k^2k^2b^2c_s^2 \right] - \lambda_H\omega^2k^2b^2 \left( \omega^2 - k^2c_s^2 \right) = 0
\]

\[ \Rightarrow \] waves now dispersive, $\lambda_H \equiv (k\delta_i)^2$ Hall parameter
Hameiri et al, PoP 12, 072109 (2005): study DR, vary $\sigma = \frac{c_s^2}{b^2}$

$\Rightarrow$ wave normal (phase) and ray surfaces (group)

$\Rightarrow$ still 3 pairs of waves (forward-backward)

$\Rightarrow$ all 3 waves dispersive: seen in $\omega - k$ diagrams
• $\omega - k$ diagrams can be shown for varying $\vartheta$ (angle $k$ and $B$)

⇒ from parallel to perpendicular
alternative representation: phase diagrams (wave normal surfaces): fix $\lambda_H = k\delta_i$, show all angles (left panel is MHD)
can show this for varying $\sigma$ (i.e. $\beta$) and animate for varying wavelength (increasing Hall parameter)
• Much more intriguing: ray surfaces (group diagrams): implicit derivation on DR yields $\frac{\partial \omega}{\partial k}$ expressions as

$$\frac{\partial \omega}{\partial k} = f_b(\omega, \sigma, k, \cos \vartheta) \hat{b} + f_n(\omega, \sigma, k, \cos \vartheta) \hat{n}$$

⇒ quantifies approximate wave fronts, $\hat{b} = \mathbf{B}/B$ and $\hat{n} = \mathbf{k}/k$
for $\sigma = 0.5$, MHD to large Hall parameter
$\Rightarrow$ note the sometimes ‘strange’ ordering (slow-Alfvén-fast)
• for $\sigma = 1$, MHD (also special in MHD!) to large Hall parameter
for $\sigma = 2$, MHD to large Hall parameter animation (increasing Hall parameter)
• relevant as test for numerical Hall-MHD: Porth et al, ApJS 214, 2014 (MPI-AMRVAC): pressure pattern emerging from interference, fast & Alfvén envelope
Ideal 2-fluid diagrams

  ⇒ DR best written in terms of $\bar{\omega} = \omega/\omega_p$, $\bar{k} = kc/\omega_p = k\delta$
  with plasma frequency $\omega_p$ and skin depth $\delta$, then obtain 12-th order polynomial (6th order in $\omega^2$, fourth in $k^2$)
  ⇒ parameters $E = \Omega_e/\omega_p$ (electron cyclotron), $v = v_e/c$, $w = v_i/c$ (sound speeds) and $\mu = Zm_e/m_i$ (mass ratio)

• known limits:
  ⇒ short $k$: MHD and plasma cut-offs
  ⇒ large $k$: 2xEM ($kc$), ion and $e$ sound, $e$ and ion cycl. res.
- $\omega-k$ diagrams, for varying angles $\cos(\vartheta) = k \cdot B / kB$, coronal loop
• vary angles $\cos(\vartheta) = \mathbf{k} \cdot \mathbf{B} / kB$, coronal loop
as angle varies: branches show (avoided) crossings, ‘labeling’ waves must ultimately involve the way eigenfunctions remain similar on various branches!

**see changeover through zeros of derivative of DR**
Show alternative wave normal (phase) diagrams, for varying $k\delta$
⇒ note several branches with superluminal phase speeds!

⇒ animate through wavenumber range
similar obtain the ray (group) diagrams, again implicit derivation on 12th order DR, with limits short (EM) and long wavelength (MHD), group speeds $< c$! animate through wavenumber
suppose you resolve up to $k\delta = 10$, interference leads to:
• MHD to Hall-MHD to 2-fluid model: increasing complexity in wave dynamics: *dispersion rules*, enormous differences in wave propagation characteristics
  \[ \Rightarrow \text{regime } k\delta \sim \mathcal{O}(0.1 - 10): \text{fascinating constructive-destructive interferences} \]

• can study all limits of physical relevance:
  \[ \Rightarrow \text{cold plasmas, electron-positron mixtures, \ldots} \]
  \[ \Rightarrow \text{limit to Hall-MHD from 2-fluid: take } \mu \to 0, \ c \to \infty \]
  \[ \Rightarrow \text{MHD as non-dispersive, long wavelength limit} \]