Coronal Seismology Application of the 3 Levels of Bayesian inference

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Purpose & Outline

We discuss the application of the 3 levels of Bayesian inference to the determination of the cross-field density structuring of coronal waveguides

Outline

1. Cross-field density structure of waveguides from transverse waves
2. Parameter inference
3. Model comparison
4. Model averaging
Transverse Coronal Loop Oscillations

Aschwanden+ (99); Nakariakov+ (99); Aschwanden+ (02); Schrijver+ (02), Verwichte+ (04); Van Doorsselaere+ (08); Erdelyi & Taroyan (08) ... White & Verwichte (12)

EUV imaging of damped transverse oscillations

Nakariakov et al. (1999)

Periods ~ 2-11 mins; damping times ~ 3-21 mins

Wave damping directly observed and can be “measured”
Resonant absorption is the most popular mechanism to explain the damping
Physical model for resonant damping in 1D

Goossens et al. (2002); Ruderman & Roberts (2002); Van Doorsselaere et al. (2004); Terradas et al. (2006)

- Static equilibrium
- No gravity
- Pressure-less flux tube
- Uniform magnetic field
- 1D density enhancement
- Linear m=1 mode (kink)
- Transverse displacement of the tube

<table>
<thead>
<tr>
<th>Mean radius</th>
<th>Radial inhomogeneity</th>
<th>Internal density</th>
<th>External density</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>$l/R$</td>
<td>$\rho_i$</td>
<td>$\rho_e$</td>
</tr>
</tbody>
</table>

\[ \rho \]

\[ R \]

\[ l \]

\[ R \]

\[ l \]
Resonance

The key is transverse non-uniformity

- Kink mode oscillations with smooth transition at tube boundary

- Resonance occurs at the position where the kink frequency matches Alfvén continuum frequency. This leads to the damping of the kink mode

- Transfer of energy from large scale motions to localized small scale motions (Alfvén waves) due to inhomogeneity in the radial direction
The damping formula

Sakurai (1991); Goossens et al. (1995); Tirry & Goossens (1996); Ruderman & Roberts (2002)

Analytical expression for the period and damping by resonant absorption can be obtained under the thin tube and thin boundary approximations (R/L << 1; l/R << 1)

\[
P = \tau_{Ai} \sqrt{2} \left( \frac{\zeta + 1}{\zeta} \right)^{1/2}
\]

\[
\frac{\tau_d}{P} = F \frac{R}{l} \frac{\zeta + 1}{\zeta - 1}
\]

F numerical factor depends on the radial density profile

Relevant parameters

- Alfvén travel time \( \tau_{Ai} \)
- Density contrast \( \zeta = \frac{\rho_i}{\rho_e} \)
- Transverse inhomogeneity length-scale \( \frac{l}{R} \)
Alternative density models

\[ \zeta = \frac{\rho_i}{\rho_e} = 10 \]

S: sinusoidal \hspace{1cm} L: linear \hspace{1cm} P: parabolic

\[ l = \frac{R}{0.25} \hspace{1cm} l = \frac{R}{0.5} \hspace{1cm} l = \frac{R}{1} \hspace{1cm} l = \frac{R}{2} \]
Classic Inversion Result

Inversion of: density contrast, transverse inhomogeneity, and Alfvén travel time, using observed period and damping

Arregui et al. (2007)

Soler et al. (2014)

1D solution space for loop models that reproduce observations

Inversion for 3 alternative density models seems to lead to significant differences
Bayesian Data Analysis

Probabilistic Inference considers the inversion problem as the task of estimating the degree of belief in statements about parameter values/model evidence

Bayes’ Rule (Bayes & Price 1763)

\[
p(\theta|D, M) = \frac{p(D|\theta, M) p(\theta|M)}{\int d\theta p(D|\theta, M) p(\theta|M)}
\]

- \(p(\theta|D, M)\) Posterior
- \(p(\theta|M)\) Prior
- \(p(D|\theta, M)\) Likelihood function
- \(\int d\theta p(D|\theta, M) p(\theta|M)\) Evidence

State of knowledge is a combination of what is known a priori independently of data and the likelihood of obtaining a data realisation actually observed as a function of the parameter vector

<table>
<thead>
<tr>
<th>Parameter Inference</th>
<th>Model Comparison</th>
<th>Model Averaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compute posterior for different combinations of parameters</td>
<td>Compare one model against other</td>
<td>Posteriors weighted with model evidence</td>
</tr>
<tr>
<td>Marginalise</td>
<td>Posterior ratios</td>
<td>Weighted posterior</td>
</tr>
</tbody>
</table>

\[
p(\theta|d) = \int p(\theta|d) d\theta_1 \ldots d\theta_{i-1} d\theta_{i+1} \ldots d\theta_N
\]

\[
\frac{p(M_i|d)}{p(M_j|d)} = \frac{p(d|M_i) p(M_i)}{p(d|M_j) p(M_j)}
\]

\[
p(\theta|d) = \sum_{i=1}^{N} p(\theta|d, M_i)p(M_i|d)
\]
Parameter inference

Infer the unknown physical parameters from observed oscillation properties:

Bayes Theorem

\[
p(\{\tau_{Ai}, \zeta, l/R\}|\{P, \tau_d\}, M) \propto p(\{p, \tau_d\}|\{\tau_{Ai}, \zeta, l/R\}, M)p(\{\tau_{Ai}, \zeta, l/R\}, M)
\]

Marginalise

\[
p(\tau_{Ai}|\{P, \tau_d\}, M) = \int p(\{\tau_{Ai}, \zeta, l/R\}|\{P, \tau_d\}, M) d\zeta d(l/R)
\]

\[
p(\zeta|\{P, \tau_d\}, M) = \int p(\{\tau_{Ai}, \zeta, l/R\}|\{P, \tau_d\}, M) d\tau_{Ai} d(l/R)
\]

\[
p(l/R|\{P, \tau_d\}, M) = \int p(\{\tau_{Ai}, \zeta, l/R\}|\{P, \tau_d\}, M) d\tau_{Ai} d\zeta
\]
What we really do

Conditional probability and marginal posteriors

Arregui & Asensio Ramos (2014)

\[ c = a \cdot b \] Joint probability of \( a \) and \( b \), given \( c \) \( p(a,b \mid c) \)

\[ p(b \mid a,c) \] probability of \( b \), given \( a \) and \( c \)

\[ p(a \mid b,c) \] probability of \( a \), given \( b \) and \( c \)

\[ p(b \mid c) \] probability of \( b \), given \( c \)

\[ p(a \mid c) \] probability of \( a \), given \( c \)

All animals are equal, but some animals are more equal than others

George Orwell, Animal Farm (1945)
Solid: TTTB approximations - Dashed: numerical

Sinusoidal

Linear

Parabolic
Solid: sinusoidal - Dotted: linear - Dashed parabolic

TTTB

Numerical
Table 1
Summary of inference results for the case with weak damping for the three alternative models by employing the TTTB and numerical forward solutions. $P = 272$ (s); $\tau_{d} = 849$ (s); $\sigma_{P} = \sigma_{\tau_{d}} = 30$ (s).

<table>
<thead>
<tr>
<th>$\theta_i$</th>
<th>TTTB approximation</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M^S$</td>
<td>$M^L$</td>
</tr>
<tr>
<td>$\tau_{Ai}$</td>
<td>171+22$^{-24}$</td>
<td>169+24$^{-24}$</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>4.4+3.7$^{-2.6}$</td>
<td>4.1+3.9$^{-2.2}$</td>
</tr>
<tr>
<td>$l/R$</td>
<td>0.3+0.4$^{-0.1}$</td>
<td>0.2+0.4$^{-0.1}$</td>
</tr>
</tbody>
</table>

The adopted density model does not seem to influence that much the inference result.

Numerical results lead to basically the same conclusion.
Model comparison

Compare the plausibility of alternative models to explain observed data

Posterior ratio for two competing models

\[
\frac{p(M_i|d)}{p(M_j|d)} = \frac{p(d|M_i) p(M_i)}{p(d|M_j) p(M_j)}
\]

A priori equally probable models > Bayes factors

\[
BF_{ij} = \frac{p(d|M_i)}{p(d|M_j)}
\]

Quantitative model comparison: compute Bayes factors as a function of measured period and damping time and use Jeffreys’ scale [Jeffreys(61); Kass & Raftery(95)]

<table>
<thead>
<tr>
<th>2 loge BF</th>
<th>Evidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-2</td>
<td>Not worth more than a bare mention</td>
</tr>
<tr>
<td>2-6</td>
<td>Positive Evidence (PE)</td>
</tr>
<tr>
<td>6-10</td>
<td>Strong Evidence (SE)</td>
</tr>
<tr>
<td>&gt; 10</td>
<td>Very Strong Evidence (VSE)</td>
</tr>
</tbody>
</table>
Example model evidence

\[ p(M_S|d) \]

linear vs. sinusoidal

\[ BF_{LS} \]

parabolic vs. sinusoidal

\[ BF_{PS} \]

linear vs. parabolic

\[ BF_{LP} \]
Model averaging

Weight posteriors from alternative models with the relative evidence for each one

Model-averaged posterior for parameter $\theta$ is

$$p(\theta|d) = \sum_{i=1}^{N} p(\theta|d, M_i)p(M_i|d) = p(M_1|d) \sum_{i=1}^{N} B_{i1}p(\theta|d, M_i)$$

Take e.g., model $M_1$ as reference model and compute Bayes factors with respect to it

Further assume that prior probabilities for the $N$ models are all equal $p(M_i)=1/N$

With these assumptions, the posterior for the reference model $M_1$ is given by

$$p(M_1|d) = \frac{1}{1 + \sum_{i=2}^{N} B_{i1}}$$
Model averaging result - case 1

Solid: sinusoidal - Dashed: linear - Dotted: parabolic - symbols: averaged posterior
Model averaging result - case 2

Solid: sinusoidal - Dashed: linear - Dotted: parabolic - symbols: averaged posterior
Conclusions

We have applied the three levels of Bayesian inference to the problem of obtaining information on the density structuring in coronal waveguides from damped transverse oscillations.

Three alternative models (sinusoidal, linear, parabolic) for the density profile at the non-uniform boundary layer.

In spite of the differences from the classic inversion using the alternative models, Bayesian inference led to very similar results.

The application of model comparison techniques could enable us to differentiate between the most plausible model in view of observed period and damping time, but only for strongly damped oscillation.

Nevertheless, the three models have different levels of evidence and, even if this evidence is not overwhelming, Bayesian model averaging can be applied to obtain a combined posterior for the unknown parameters that includes the individual model evidence.