Bayesian Inference and Model Comparison for Solar Atmospheric Seismology

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in collaboration with
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We present three recent results from the application of Bayesian analysis techniques to solar atmospheric seismology:

- Alfvén speed and transverse density inhomogeneity in coronal loops
- Cross-field density structuring from spatial damping of transverse oscillations
- Bayesian inference and model comparison from multiple harmonic oscillations
Confronting observations and theory to infer physical parameters is not an easy task.

We face aspects that are fundamentally different:

Seismology involves the solution of **two different problems**

- **Forward problem**
  - **Cause**: Theoretical models and parameters
  - **Consequences**: Theoretical wave properties

- **Inverse problem**
  - **Cause**: Unknown physical models/parameters
  - **Consequences**: Observed wave properties

A meaningful approach to inverse problems exists since long time ago in the Bayesian framework which considers “Probability as extended logic” (E.T. Jaynes).
**A one-slide introduction to probability**

**What is probability:** Probability quantifies randomness and **uncertainty**

**What is statistics:** Statistics uses probability to make scientific inferences

**Use of probability:** There are two main schools / lines of though / religions

<table>
<thead>
<tr>
<th></th>
<th><strong>Frequentists</strong></th>
<th></th>
<th><strong>Bayesians</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Interpretation of probability</strong></td>
<td>Measure frequencies</td>
<td>Long-run relative frequency in the limit of infinite repetitions</td>
<td>Measure of degree to which a given proposition is supported by data</td>
</tr>
<tr>
<td><strong>Focus on</strong></td>
<td>Alternative data: compare probs. of different data realizations</td>
<td>Alternative hypotheses: compare probs. of different hypotheses <strong>in view of data</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Useful for</strong></td>
<td>Counting Characterizing data</td>
<td>Inference and model comparison</td>
<td></td>
</tr>
</tbody>
</table>

**Astrophysics observational science > data are fixed!**
The Bayesian framework defines rigorous tools to perform inference and model comparison by looking at how data constrain parameters/models.
Bayes’ theorem (Bayes & Price 1763)

Mathematical formulation of “the process of learning from experience”

\[
p(\theta|D, M) = \frac{p(D|\theta, M)p(\theta|M)}{\int d\theta p(D|\theta, M)p(\theta|M)}
\]

Posterior

\[
p(\theta|D, M) = \frac{p(D|\theta, M)p(\theta|M)}{\int d\theta p(D|\theta, M)p(\theta|M)}
\]

Prior

\[
\int d\theta p(D|\theta, M)p(\theta|M)
\]

Likelihood function

\[
\int d\theta p(D|\theta, M)p(\theta|M)
\]

Evidence

State of knowledge is a combination of what is known a priori independently of data and the likelihood of obtaining a data realization actually observed as a function of the parameter vector

Parameter Inference

Compute posterior for different combinations of parameters

Model Comparison

Compare one model against other by computing posterior ratios

\[
\frac{p(M_i|d)}{p(M_j|d)} = \frac{p(d|M_i) p(M_i)}{p(d|M_j) p(M_j)}
\]

Marginalise

\[
p(\theta_i|d) = \int p(\theta|d) d\theta_1 \ldots d\theta_{i-1} d\theta_{i+1} \ldots d\theta_N
\]
Example #1
Coronal loop oscillations

Aschwanden et al. (1999); Nakariakov et al. (1999); Aschwanden et al. (2002); Schrijver et al. (2002), Verwichte et al. (2004) ... White & Verwichte (2012)

• Transverse standing MHD kink mode of a magnetic flux tube - lateral displacement of the tube (Nakariakov99)- Multiple harmonics (Verwichte04)

• Resonant damping - coupling of global motion to local Alfvén waves (Hollweg & Yang88; Goossens02)

Periods ~ 2-11 mins

Damping times ~ 3-21 mins
**Forward problem**

- Thin tube approximation for the period (Edwin & Roberts 1983)

\[ P = \tau_{Ai} \sqrt{2} \left( \frac{\zeta + 1}{\zeta} \right)^{1/2} \]

- Thin boundary approximation for the damping (Goossens et al. 1992; Ruderman & Roberts 2002)

\[ \frac{\tau_d}{P} = \frac{2}{\pi} \frac{\zeta + 1}{\zeta - 1} \frac{1}{l/R} \]

**Inverse problem**

- Observed periods and damping times can be reproduced by infinite number of models

- But they must follow a particular 1D solution space (Arregui et al. 2007)

\[ P(\zeta, l/R, \tau_{Ai}) = P_{obs} \]

\[ \frac{\tau_d}{P}(\zeta, l/R) = \left( \frac{\tau_d}{P} \right)_{obs} \]

3 parameters \( (\zeta, l/R, \tau_{Ai}) \)

2 observables \( (P, \tau_d/P) \)
Inversion in coronal loops

Analytic/numerical inversion schemes

**Advantages**

- No assumption on particular values for parameters
- General solution from which limiting cases can be studied

**Limitations**

- Infinite number of equally valid solutions
- No clear way to propagate errors from observations to inferred quantities

Alfvén speed constrained to a narrow range

Excellent analytic/numerical agreement
Bayesian inversion

Prior information

Density contrast:
3 different options

Inhomogeneity length-scale:
Uniform in range 0-2

Alfvén travel time:
Uniform in range determined by period

Likelihood function

\[
p(d|\theta) = \left(2\pi \sigma_P \sigma_\tau\right)^{-1} \exp \left\{ \frac{[P - P^{\text{syn}}(\theta)]^2}{2\sigma_P^2} + \frac{[\tau_d - \tau_d^{\text{syn}}(\theta)]^2}{2\sigma_\tau^2} \right\}
\]

Synthetic data from forward problem

\[
\sigma_P^2 \quad \sigma_\tau^2
\]

Variances associated to period and damping time
Optimal results are obtained with density information

Suppose we have some information on densities

Arregui & Asensio Ramos (2011)

Prior information

Density contrast:
Gaussian centered in $\zeta = 5$, $\sigma_\zeta = 0.1\zeta$

Inhomogeneity
Uniform in $l/R \in [0 - 2]$

Alfvén travel time
Uniform in $\tau_{Ai} \in [1 - 400]$

Data is able to constrain the problem

\[
\begin{align*}
P &= 232 \text{ s} \\
\tau_d/P &= 3.8
\end{align*}
\]
Marginal posteriors

All parameters of interest **fully constrained** when info on density inserted
Table 2. Analytic (A) and Bayesian (B) inversion results for the analyzed loop oscillation events.

<table>
<thead>
<tr>
<th>Oscillation properties</th>
<th>Inversion results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytic</td>
</tr>
<tr>
<td></td>
<td>(\tau_{d} (s))</td>
</tr>
<tr>
<td>#</td>
<td>(P (s))</td>
</tr>
<tr>
<td>1</td>
<td>261</td>
</tr>
<tr>
<td>2</td>
<td>265</td>
</tr>
<tr>
<td>3</td>
<td>316</td>
</tr>
<tr>
<td>4</td>
<td>277</td>
</tr>
<tr>
<td>5</td>
<td>272</td>
</tr>
<tr>
<td>6</td>
<td>522</td>
</tr>
<tr>
<td>7</td>
<td>435</td>
</tr>
<tr>
<td>8</td>
<td>143</td>
</tr>
<tr>
<td>9</td>
<td>423</td>
</tr>
<tr>
<td>10</td>
<td>185</td>
</tr>
<tr>
<td>11</td>
<td>390</td>
</tr>
</tbody>
</table>

*Inversions with Gaussian prior use contrast estimates by Aschwanden et al. (2003)*
Example #2
Spatially damped transverse coronal waves

Coronal waves
Tomczyk et al. (2007); Tomczyk & McIntosh (2009)

6. Conclusions and discussion

5. Resistive calculations

approximations are not used. The resistive calculation applies to
there are no implicit assumptions about the TT or TB approxi-
mations that we have already derived are quite satisfactory,
this position. This explains the small deviations from the resis-
ting to spatial damping. By including resistivity, the eigenvalue
are fully non-uniform. We have solved the eigenvalue problem
in the range 0–4 mHz, which can be explained by the theory
of resonant absorption by analyzing obser-
pressions are the result of assuming that
have a thin non-uniform layer or
scale density inhomogeneities as short loops. This has also
standing kink waves in individual coronal loops observed by
established that in the solar corona there are ubiquitous propa-
gating mechanism of resonant absorption. That overtones of standing
power loss depending on the properties of the equilibrium, in
the variation in the local Alfvén speed. With respect to mode
and
phase speeds of the kink MHD waves. This allows us to trans-
late the results from the temporally damped waves (along the waveguide) to accommodate the kink mode in the
atmosphere.

5. CONCLUSIONS

Different inward/outward power

Resonant absorption favours low-f waves

Selective spatial damping

Terradas, Goossens, Verth (2010)

\[ L_d \sim \frac{1}{f} \]

Frequency dependence

Verth et al. (2010)
Spatial damping of propagating kink waves
Terradas Goossens & Verth (2010) see also Soler et al. (2011a,b) Pascoe, Wright, De Moortel (2010)

For propagating transverse kink waves resonant absorption produces spatial damping
Two damping regimes


The decay of resonantly damped kink oscillations shows 2 distinct regimes:
Initial Gaussian decay + subsequent exponential damping

Additional information without the need to include new parameters

Gaussian damping

\[ \frac{L_g}{\lambda} = \left( \frac{2}{\pi} \right) \left( \frac{R}{l} \right)^{1/2} \left( \frac{\zeta + 1}{\zeta - 1} \right) \]

Exponential damping

\[ \frac{L_d}{\lambda} = \left( \frac{2}{\pi} \right)^2 \left( \frac{R}{l} \right) \left( \frac{\zeta + 1}{\zeta - 1} \right) \]

Regime change at location

\[ h = \frac{L_g^2}{L_d} = \lambda \left( \frac{\zeta + 1}{\zeta - 1} \right) \]
Bayesian inference with propagating waves


Inversion of density contrast and transverse inhomogeneity length scale using Gaussian damping length and height of change of damping regime as data

Generate synthetic data using analytical forward model

\[
\frac{L_g}{\lambda} = \left(\frac{2}{\pi}\right) \left(\frac{R}{l}\right)^{1/2} \left(\frac{\zeta + 1}{\zeta - 1}\right) \\
h = \frac{L_g}{L_d} = \lambda \left(\frac{\zeta + 1}{\zeta - 1}\right)
\]

Parameter space

\[\zeta = 1.5, 2, 3, 4\] and \[l/R = 0.05, 0.15, 0.2, 0.4\]

Likelihood + uniform priors for contrast and length scale

\[
p(d|\theta) = \left(2\pi\sigma_{L_g}\sigma_h\right)^{-1} \exp\left\{\frac{\left[L_g - L_g^{\text{syn}}(\theta)\right]^2}{2\sigma_{L_g}^2} + \frac{[h - h^{\text{syn}}(\theta)]^2}{2\sigma_h^2}\right\} \quad p(\theta_i) = \frac{1}{\theta_i^{\text{max}} - \theta_i^{\text{min}}} \quad \text{for} \quad \theta_i^{\text{min}} \leq \theta \leq \theta_i^{\text{max}}
\]

Use Bayes’ rule and marginalise

\[
p(\theta|D, M) = \frac{p(D|\theta, M)p(\theta|M)}{\int d\theta p(D|\theta, M)p(\theta|M)} \quad p(\theta_i|d) = \int p(\theta|d)d\theta_1 \ldots d\theta_{i-1}d\theta_{i+1} \ldots d\theta_N
\]
Inversion result - example

The existence of two damping regimes enables us to constrain the two parameters of interest.

They fully determine the cross-field density structuring.
Inversion results

Inversion with analytical forward model

Table 1. Inversion of Synthetic Data Using the Analytical Forward Model

<table>
<thead>
<tr>
<th>Synthetic Parameters</th>
<th>Synthetic Data</th>
<th>Inversion Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>$l/R$</td>
<td>$L_g/\lambda$</td>
</tr>
<tr>
<td>1.5</td>
<td>0.05</td>
<td>14.2</td>
</tr>
<tr>
<td>1.5</td>
<td>0.15</td>
<td>8.2</td>
</tr>
<tr>
<td>1.5</td>
<td>0.2</td>
<td>7.1</td>
</tr>
<tr>
<td>1.5</td>
<td>0.4</td>
<td>5.0</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>5.7</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>3.3</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>2.9</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
<td>2.0</td>
</tr>
<tr>
<td>4</td>
<td>0.05</td>
<td>4.8</td>
</tr>
<tr>
<td>4</td>
<td>0.15</td>
<td>2.7</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>2.4</td>
</tr>
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<td>1.7</td>
</tr>
<tr>
<td>10</td>
<td>0.5</td>
<td>1.1</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>0.8</td>
</tr>
<tr>
<td>10</td>
<td>1.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Inversion technique correctly recovers input parameters

Analytical forward model accurate enough when compared to simulation inversions

Large density contrasts represent a challenge from observational point of view

Inversion with numerical simulation

Table 2. Inversion of Numerical Data From Simulations

<table>
<thead>
<tr>
<th>Simulation Parameters</th>
<th>Fitted Data</th>
<th>Inversion Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>$l/R$</td>
<td>$L_g/\lambda$</td>
</tr>
<tr>
<td>1.5</td>
<td>0.05</td>
<td>11.5</td>
</tr>
<tr>
<td>1.5</td>
<td>0.15</td>
<td>7.9</td>
</tr>
<tr>
<td>1.5</td>
<td>0.2</td>
<td>7.0</td>
</tr>
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<td>0.4</td>
<td>5.0</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>5.5</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
<td>3.5</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>3.1</td>
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<tr>
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<tr>
<td>4</td>
<td>0.2</td>
<td>2.7</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>2.3</td>
</tr>
</tbody>
</table>
Example #3
Multiple mode harmonic oscillations

Verwichte et al. (2004); Andries et al. (2005); Andries, Arregui, & Goossens (2005)

Observations

Detection of multiple harmonics in two coronal loops. Simultaneous presence of fundamental and first harmonic

Verwichte et al. (2004)

Theory

In a longitudinally inhomogeneous flux tube the period ratio of first overtone to fundamental mode is smaller than 2 and depends on density stratification

\[ \frac{P_1}{P_2} < 2 \]
We can mimic an exponentially stratified atmosphere using a straight tube model by projecting the vertical density variation onto a semicircular loop of length L and height L/\pi.

\[ \rho(z) = \rho_0 \exp(-z/H) \quad \rho(s) \approx \exp(-L \sin(\pi s/L)/\pi H) \]

We consider a vertically stratified atmosphere and a curved coronal loop. Use the observed period ratio together with theoretical calculations to estimate the density scale-height in the solar atmosphere.
$2 - \frac{P_1}{P_2} = 0.36 \pm 0.23 \implies \frac{H\pi}{L} : [0.276, 1.35] \text{ with an estimated value } \frac{H\pi}{L} = 0.48$

H: [20, 99] Mm with an estimated value of $H = 36$ Mm
Magnetic flux tube expansion

Verth & Erdélyi (2008); Ruderman et al. (2008); Verth et al. (2008)

Expansion of magnetic loop affects period ratio

The period ratio of first overtone to fundamental mode is larger than 2 and depends on magnetic expansion

Expansion of magnetic loop affects period ratio

The approximate equation of the tube boundary is

\[ \rho_i(z) = \rho_e(z) \]

where the prime indicates the derivative,

\[ J_0(x), \quad \text{and} \quad J_1(x) \]

are arbitrary constants and

\[ \lambda_0 = \frac{\pi}{C_1}, \quad \lambda_1 = \frac{\pi}{C_2} \]

are the roots of the Bessel functions

\[ J_0(\lambda_0 z) = 0, \quad J_1(\lambda_1 z) = 0 \]

The functional form of the magnetic field is

\[ B(r, z) = B^0 \exp \left( \frac{z}{C_0} \right) \]

for all \( z \). Then, using the approximations in Eq. (3) in V07. However, with increasing magnetic stratification is a very reasonable analytical approximation of the period ratio

\[ \frac{\omega_2}{\omega_1} \approx \frac{\lambda_2}{\lambda_1} \]

is a very reasonable analytical approximation of the frequency difference, and estimating coronal density scale heights...
Bayesian inference

Model 1: density stratification

\[ r_1 = \frac{P_1}{2P_2} = 1 - \frac{4}{5} \left( \frac{\eta}{\eta + 3\pi^2} \right) \]

\[ \eta = \frac{L}{\pi H} \]

Safari et al. (2007)

Model 2: magnetic expansion

\[ r_2 = \frac{P_1}{2P_2} = 1 + \frac{3(\Gamma^2 - 1)}{2\pi^2} \]

\[ \Gamma = \frac{r_a}{r_f} \]

Verth & Erdélyi (2008)

Parameter Inference: compute posteriors for given value of measured period ratios \( r \)

Gaussian likelihoods

\[
p(r|\eta, M_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{(r - r_1)^2}{2\sigma^2} \right]
\]

\[
p(r|\Gamma, M_2) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{(r - r_2)^2}{2\sigma^2} \right]
\]

Uniform priors

\[
p(\eta|M_1) = \frac{1}{\eta^{\text{max}} - \eta^{\text{min}}} \quad \text{for} \quad \eta^{\text{min}} \leq \eta \leq \eta^{\text{max}}
\]

\[
p(\Gamma|M_2) = \frac{1}{\Gamma^{\text{max}} - \Gamma^{\text{min}}} \quad \text{for} \quad \Gamma^{\text{min}} \leq \Gamma \leq \Gamma^{\text{max}}
\]

Marginalise

\[
p(\theta_i|d) = \int p(\theta|d)d\theta_1 \ldots d\theta_{i-1}d\theta_{i+1} \ldots d\theta_N
\]
Bayesian inference results

Well-defined posteriors. Inference with correctly propagated uncertainties from data to inferred parameters

Coronal density scale heigh: \( H \sim 21 \text{ Mm} \) and \( H \sim 56 \text{ Mm} \) (for \( L/\pi = 70 \text{ Mm} \))

Magnetic tube expansion factor: \( \Gamma \sim 1.20 \) and \( 1.87 \)
Bayesian model comparison

Assess the performance of 3 models in explaining data:

- M0 uniform loop
- M1: stratified loop
- M2 expanding loop

Marginal Likelihoods

\[ p(r | M_i) = \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} p(r, \theta | M_i) d\theta = \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} p(r | \theta, M_i) p(\theta | M_i) d\theta \]

Quantitative model comparison: compute Bayes factors as a function of measured period ratios and use Jeffreys’ scale (Jeffreys 1961; Kass & Raftery 1995)

\[ \frac{p(M_i | r)}{p(M_j | r)} = \frac{p(r | M_i)}{p(r | M_j)} \frac{p(M_i)}{p(M_j)} \]

\[ BF_{ij} = \frac{p(r | M_i)}{p(r | M_j)} \]

<table>
<thead>
<tr>
<th>2 log_e BF</th>
<th>Evidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-2</td>
<td>Not worth more than a bare mention</td>
</tr>
<tr>
<td>2-6</td>
<td>Positive Evidence (PE)</td>
</tr>
<tr>
<td>6-10</td>
<td>Strong Evidence (SE)</td>
</tr>
<tr>
<td>&gt; 10</td>
<td>Very Strong Evidence (VSE)</td>
</tr>
</tbody>
</table>
A period ratio smaller than one is not sufficient evidence for density stratification.

Level of evidence depends on data and their uncertainties.
A period ratio larger than one is not sufficient evidence for magnetic expansion

Level of evidence depends on data and their uncertainties
Different levels of evidence for density stratification and magnetic tube expansion
Summary

• Bayesian analysis tools enable us to perform parameter inference / model comparison combining observations of transverse oscillations in the corona with MHD theory results

• Parameter inference successful in determining Alfvén speed and cross-field density structuring

• Method incorporates consistently calculated confidence levels and uncertainties

• Bayesian model comparison enables us to assess quantitatively which hypothesis, among competing mechanisms, better explains data