Collisional frequencies, pressure tensor and plasma drifts

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Outline of the talk

- Introduction,
- Boltzman equation and collision term,
- Collisional frequencies,
- Three-fluid equations and the closure problem,
- Transport coefficients, general description,
- Particle drifts in a partial ionized plasma,
- Plasma drifts and pressure tensor,
- Different approaches for partial ionized plasmas,
- Summary and conclusions.

Introduction

- Partially ionized plasmas are relevant in many astrophysical situations and laboratory experiments.
- MHD theory provides a good approximation in many cases, and it is relatively simple, has many interesting mathematical properties and has been studied extensively from the computational point of view.
- However, there are situations beyond the MHD. For example, it does not consider partial ionization.
- Multi-fluid plasmas provides a better framework for understanding these processes, but no consensus on several key points.

Introduction

- Three-fluid equations: first step towards partially ionized plasmas. Only electrons, protons and neutral H (*α*=*i*, *n*, *e*), but easy to generalize for more species.

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{u}_\alpha) = S_\alpha,$$

$$\rho_\alpha \frac{D\vec{u}_\alpha}{Dt} = e_\alpha n_\alpha (\vec{E} + \vec{u}_\alpha \times \vec{B}) + \rho_\alpha \vec{g}$$

$$-\nabla \cdot \hat{\mathrm{p}}_\alpha + \vec{R}_\alpha - \vec{u}_\alpha S_\alpha,$$

$$\frac{D}{Dt}(3p_\alpha/2) + (3p_\alpha/2)(\vec{\nabla} \cdot \vec{u}_\alpha) + (\hat{\mathrm{p}}_\alpha \cdot \vec{\nabla}) \cdot \vec{u}_\alpha$$

$$+\vec{\nabla} \cdot \vec{q}_\alpha = M_\alpha - \vec{u}_\alpha \cdot \vec{R}_\alpha + \frac{1}{2} u_\alpha^2 S_\alpha = Q_\alpha$$

See Khomenko's presentation!

- Theory is not complete: some quantities are not defined.
- Some terms must be neglected or estimated, but they may have relevant physics!

Boltzman equation

- Distribution function each species in the system: contains all the relevant information.
- Boltzman equation gives the evolution. The EM fields also have a contribution of the field from the rest of particles (Vlassov fields), and contains a collisions term.

$$\frac{\partial f^\alpha}{\partial t} = -\vec{v}_\alpha \frac{\partial f^\alpha}{\partial \vec{r}_\alpha} - \frac{e_\alpha}{m_\alpha}\left(\vec{E} + \vec{v}_\alpha \times \vec{B}\right)\frac{\partial f^\alpha}{\partial \vec{v}_\alpha} + \mathcal{K}^\alpha,$$

- Moments and macroscopic variables.: fluid description.

$$\langle \psi_\alpha(\vec{r},t)\rangle = \frac{\int_V \psi(\vec{r},\vec{v},t) f_\alpha(\vec{r},\vec{v},t) d\vec{v}}{\int_V f_\alpha(\vec{r},\vec{v},t) d\vec{v}}$$

- Two problems: higher order moments and averages over the collision term.

Collision integral

- Boltzman collisional term

$$\mathcal{K}^{\alpha} = \sum_{\beta} \int_{v_1} \int_{\Omega} \{ f_{\alpha}(\vec{r}, \vec{v}', t) f_{\beta}(\vec{r}, \vec{v}_1', t) - f_{\alpha}(\vec{r}, \vec{v}, t) f_{\beta}(\vec{r}, \vec{v}_1, t) \} \, |\vec{v}_1 - \vec{v}| \, \sigma(\Omega) \, d\vec{v}_1 d\Omega$$

- Fokker-Plank approach (diffusion and dynamical friction).

$$\mathcal{K}^{\alpha} = -\frac{\partial}{\partial v_i} \left(f_{\alpha} \left\langle \Delta v_i \right\rangle_{\alpha} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(f_{\alpha} \left\langle \Delta v_i \Delta v_j \right\rangle_{\alpha} \right).$$

- Landau collision term (fully ionized plasmas)

$$\mathcal{K}^{\alpha} = \sum_{\beta} 2\pi e_{\alpha}^2 e_{\beta}^2 \ln \Lambda \int \frac{1}{m_{\alpha}} \frac{\partial G_{rs}(\vec{v}_1 - \vec{v}_2)}{\partial v_{1r}} \left(\frac{1}{m_{\alpha}} \frac{\partial}{\partial v_{1s}} - \frac{1}{m_{\beta}} \frac{\partial}{\partial v_{2s}} \right) f^{\alpha}(\vec{r}_1, \vec{v}_1, t) f^{\beta}(\vec{r}_2, \vec{v}_2, t) d\vec{v}_2$$

- Inelastic collisions not considered, only approximated expressions. For example electrons, with ionization, recombination and attachment to neutrals rates. (Bittencourt)

$$St_e = \int_V \mathcal{K}_e^{\text{inelastic}} d\vec{v} = m_e (k_i n_e - k_r n_e^2 - k_a n_e),$$

Collision frequencies

- Using the differential scattering cross section and averaging over Maxwellian distributions (Rozhansky & Tsendin)

$$\nu_{\alpha\beta}(T_\alpha, T_\beta) = \frac{16}{3}\left(\frac{2\pi}{\gamma_{\alpha\beta}}\right)^{1/2} \int_0^\infty \xi^5 e^{-\xi^2} \int_0^\pi \sigma_{\alpha\beta}\left((2/\gamma_{\alpha\beta})^{1/2}\ \xi, \chi\right)(1-cos\chi)sin\chi d\chi d\xi,$$

- With the Coulomb potential (two-particle collisions), charged particle collisions (Braginskii)

$$\nu_{ei} = \nu_{ee} = \frac{4\sqrt{2\pi}e^4}{3(4\pi\epsilon_0)^2 m_e^{1/2} k_B^{3/2}} \frac{n_e \Lambda Z_e^2}{T^{3/2}} = 3.7 \times 10^{-6} \frac{n_e \Lambda Z_e^2}{T^{3/2}}$$

$$\nu_{ii} = \frac{4\sqrt{\pi}e^4}{3(4\pi\epsilon_0)^2 m_i^{1/2} k_B^{3/2}} \frac{n_i \Lambda Z_i^4}{T^{3/2}} = 6 \times 10^{-8} \frac{n_i \Lambda Z_i^4}{T^{3/2}} = 2.6 \times 10^{-6}\left(\frac{m_e}{m_i}\right)^{1/2} \frac{n_i \Lambda Z_i^4}{T^{3/2}}$$

- Neutral collisions (Spitzer), hard-sphere

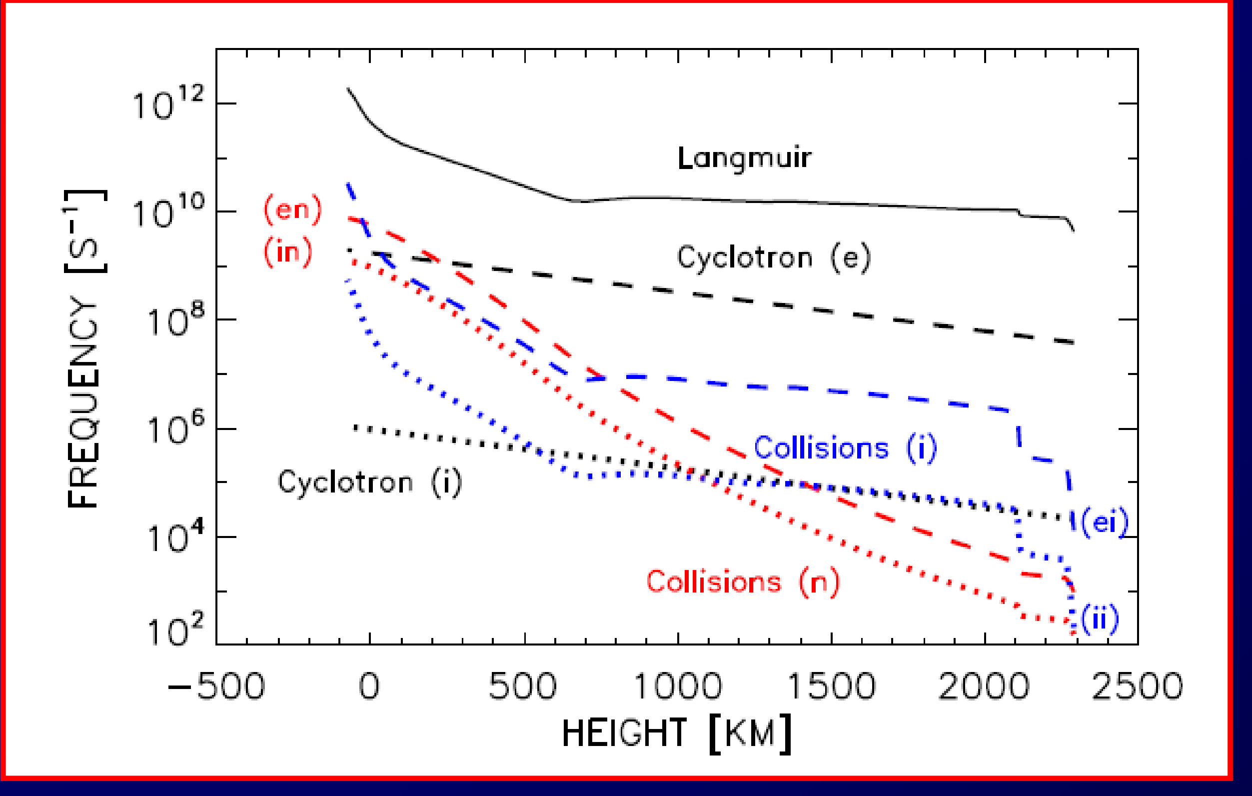
$$\nu_{in} = n_n \sqrt{\frac{8k_B T}{\pi m_{in}}} \sum_{in}$$

$$\nu_{en} = n_n \sqrt{\frac{8k_B T}{\pi m_{en}}} \sum_{en}$$

$$\sum_{in} = 5 \times 10^{-19}\ \mathrm{m}^2, \ \sum_{en} = 10^{-19}\ \mathrm{m}^2$$

Collisional frequencies

- Computed used typical values for the lower solar atmosphere.



VAL-C model

100 G at *z=0* exponentially decaying with heigh.

- Depending on the height, different terms are dominating!

Higher order moments

- Momentum and energy conservation involve higher order moments of the distribution function.
- Pressure tensor. Normally only the scalar pressure is used (one third of the trace).

$$\boxed{\hat{\mathrm{p}}_{\alpha\mathrm{jk}}(\vec{r},t) = m_\alpha n_\alpha(\vec{r},t)\int_V \vec{c}_j\vec{c}_k f_\alpha(\vec{r},\vec{v},t)d\vec{v} = \rho_\alpha\langle\vec{c}_{\alpha j}\vec{c}_{\alpha k}\rangle}$$

- Heat flux vector

$$\boxed{\vec{q}_\alpha = \frac{m_\alpha n_\alpha}{2}\int_V \vec{c}_\alpha c_\alpha^2 f_\alpha(\vec{r},\vec{v},t)d\vec{v} = \rho_\alpha\langle c_\alpha^2\vec{c}_\alpha\rangle/2}$$

- Higher order fluid equations might be obtained for these quantities, but involve even higher order moments (and more complicated averages over collisional terms)

Fluid equations

- Taking the momentums of Boltzman equations up to second order (and leaving the collisional terms unspecified).
- System of equations for the hydrodynamical variables of each species (ρ_α, $\boldsymbol{u}_\alpha$, p_α) and the electrodynamic variables ($\boldsymbol{B}$, $\boldsymbol{E}$).
- System not closed!
  - Inelastic collisions
  - Friction terms
  - Collision heat terms
  - Pressure tensors
  - Heat flux vectors

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \vec{u}_\alpha) = S_\alpha,$$

$$\rho_\alpha \frac{D\vec{u}_\alpha}{Dt} = e_\alpha n_\alpha (\vec{E} + \vec{u}_\alpha \times \vec{B}) + \rho_\alpha \vec{g} - \nabla \cdot \hat{\mathrm{p}}_\alpha + \vec{R}_\alpha - \vec{u}_\alpha S_\alpha,$$

$$\frac{D}{Dt}(3p_\alpha/2) + (3p_\alpha/2)(\vec{\nabla} \cdot \vec{u}_\alpha) + (\hat{\mathrm{p}}_\alpha \cdot \vec{\nabla}) \cdot \vec{u}_\alpha + \vec{\nabla} \cdot \vec{q}_\alpha = M_\alpha - \vec{u}_\alpha \cdot \vec{R}_\alpha + \frac{1}{2} u_\alpha^2 S_\alpha = Q_\alpha$$

Transport theory

- Objective: achieve closure of the fluid equations by relating the unknown fluxes to the forces (hydrodynamical and electrodynamic varibles).
- Quasi-local thermodynamical equilibrium assumption: the state of the system is locally determined by a Maxwellian function plus a small correction term (which is a function of the equilibrium plasma parameters).

$$\boxed{f_\alpha = f_\alpha^0 + f_\alpha^1 = \frac{n_\alpha}{(2\pi k_B T_\alpha / m_\alpha)^{3/2}} \exp\left(-\frac{m_\alpha(\vec{v} - \vec{u}_\alpha)^2}{2k_B T_\alpha} \right) + f_\alpha^1.}$$

- All the unknown fluxes can be expressed by obtaining approximations to the correction (Boltzman equation), namely the departures of the thermodyamical equilibrium: the temperature and velocity gradients and the temperature difference and velocity difference between species.

Transport coefficients

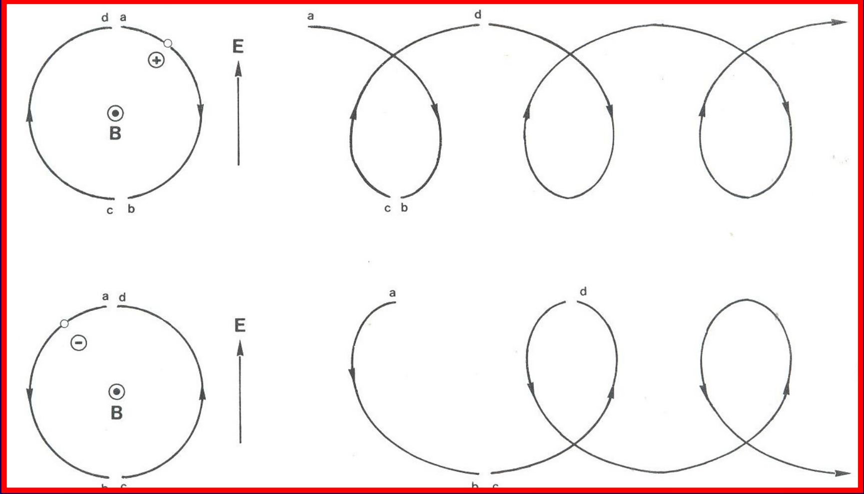
- Friction force, $\vec{R}^{\vec{u}}_{\alpha} = \hat{c}^{\vec{u}}_{\alpha\beta} \nu_{\alpha\beta} \mu_{\alpha\beta} n_{\alpha} (\vec{u}_{\alpha} - \vec{u}_{\beta}),$
- Thermal force, $\vec{R}^{T}_{\alpha\beta} = \hat{C}^{T}_{\alpha\beta} n_{\alpha} \nabla T_{\alpha},$
- Heat conductivity, $\vec{q}^{T}_{\alpha} = -\hat{\kappa}_{\alpha} \nabla T_{\alpha},$
- Heat due convection , $\vec{q}^{\vec{u}}_{\alpha} = \sum_{\beta} \hat{C}^{T}_{\alpha\beta} n_{\alpha} (\vec{u}_{\alpha} - \vec{u}_{\beta}),$
- Collisional heat production, $Q_{\alpha} = Q^{\Delta}_{\alpha} + Q^{\vec{v}}_{\alpha} = \sum_{\beta=i,n} \frac{3m_e}{m_{\beta}} n_e \nu_{e\beta} (T_{\beta} - T_e) - \sum_{\beta=i,n} \vec{R}^{\vec{v}}_{\alpha\beta} (\vec{v}_{\alpha} - \vec{v}_{\beta}),$
- Viscosity, $\hat{\mathrm{p}}_{\alpha\mathbf{jk}}(\vec{r},t) = \vec{p}_{\alpha jj}(\vec{r},t)\delta_{jk} + \hat{\pi}_{\alpha\mathbf{jk}}(\vec{r},t),$ $\hat{\pi}_{\alpha} = \hat{\eta}_{\alpha} : \hat{W}_{\alpha}, \quad \hat{W}_{jk} = \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} - \frac{2}{3}\delta_{jk} \nabla \vec{v},$
- Mobility, conductivity, diffusion and thermodiffusion. $\vec{\Gamma}_{\alpha} = n_{\alpha} \vec{u}_{\alpha} = -Z_{\alpha} \hat{b}_{\alpha} \vec{E} - \hat{D}_{\alpha} \nabla n_{\alpha} - \hat{D}^{T}_{\alpha} \nabla T_{\alpha},$
- Elementary theory (collisional frequencies independent of v): no thermal force (or heat), unity tensors and only friction heat.
- Magnetized plasma: non diagonal terms (drifts!).

Plasma drifts

- Movement of particles under uniform electromagnetic fields.

$$\vec{r}(t) = \left(r_{\parallel}^{0} + v_{\parallel}^{0} t + \frac{e}{2m} E_{\parallel}^{2} t^{2}\right) \vec{b} + \left(r_{\perp}^{0} \vec{n}_1 + \frac{v_{\perp}^{0}}{\omega_c} \vec{n}_2(\phi_0 + \omega_c t)\right) + \frac{1}{B^2}[\vec{E} \times \vec{B}]t.$$

- Uniform field:
  - electric force,
  - Larmor or cyclotron giration, (different sign for + and – charges).
  - electric drift, same sign for all charges, friction with neutrals (ambipolar).



$$\vec{v}_E = \frac{1}{B^2}\vec{E} \times \vec{B}.$$

Plasma drifts

- Non-uniform EM fields (guiding center approximation, Morozov & Solov'ev, Balescu).

$$\frac{\partial Y}{\partial t} = \left(v_{\parallel} + \frac{v_{\perp}^2}{2\omega_c}\vec{b}\cdot\left(\nabla\times\vec{b}\right)\right)\vec{b} + \frac{\vec{E}\times\vec{B}}{B^2} + \frac{v_{\perp}^2}{2\omega_c B}b\times(\nabla B) + \frac{v_{\parallel}^2}{\omega_c}\vec{b}\times(\vec{b}\cdot\nabla)\vec{b},$$

- Non-uniform field: even more types of drifts:
  - grad-B drift,
  - centrifugal drift,
  - External force drift (ex. gravity).

- In one-fluid approximation these terms are included in the generalized Ohm's law and induction equation

$$\left[\vec{E} + \vec{u}\times\vec{B}\right] = \eta\mu\vec{J} + \eta_H\frac{\mu}{|B|}\left[\vec{J}\times\vec{B}\right] - \eta_A\frac{\mu}{|B|^2}\left[(\vec{J}\times\vec{B})\times\vec{B}\right]$$

$$\vec{v}_{\perp} = \frac{\vec{E}\times\vec{B}}{B^2} - \frac{1}{\sigma B^2}\vec{j}\times\vec{B} - \frac{\eta_H}{B^3}\left(\vec{j}\times\vec{B}\right)\times\vec{B} - \frac{\eta_A}{B^4}\left[\left(\vec{j}\times\vec{B}\right)\times\vec{B}\right]\times\vec{B} + \ldots$$

Transport coefficients and drifts

- These drifts affect the deviations from unitary tensors in the transport coefficients.
- For example, the mobility and conductivity tensors (Hall and Pedersen components), neglecting the inertial terms (no momentum equation):

$$\hat{\sigma} = en_e \left(\hat{b}_e + \hat{b}_i\right)$$

$$\hat{b}_e = \begin{pmatrix} b_{e\perp} & -b_{e\wedge} & 0 \\ b_{e\wedge} & b_{e\perp} & 0 \\ 0 & 0 & b_{e\|} \end{pmatrix}, \quad \hat{b}_i = \begin{pmatrix} b_{i\perp} & b_{i\wedge} & 0 \\ -b_{i\wedge} & b_{i\perp} & 0 \\ 0 & 0 & b_{i\|} \end{pmatrix},$$

$$b_{\alpha\|} = \frac{e}{\mu_{\alpha n}\nu_{\alpha n}}, \quad b_{\alpha\perp} = \frac{e\nu_{\alpha n}}{\mu_{\alpha n}\omega_{c\alpha}^2}, \quad b_{\alpha\wedge} = \frac{e}{\mu_{\alpha n}\omega_{c\alpha}}.$$

- If cyclotron frequencies are larger than the collisional frequencies these effects are small.
- Can these kind of expressions be plugged in the fluid equations?

The pressure tensor

- Non-isotropic parts of the pressure tensor can be related to different kinetic temperatures in the spatial directions.
- In the presence of an uniform magnetic field, the pressure tensor is anisotropic (Chew, Goldberg & Low), but still only diagonal terms

$$\boxed{\hat{\pi}_{ij} = (P_{\alpha\|} - P_{\alpha\perp})(b_i b_j - \delta_{i,j}/3)}$$

- If non-diagonal components of the pressure tensor are neglected, drift effects are not fully taken into account!
- In non-uniform magnetic fields there is another effect: the cyclotron movement of positive and negative particles are in opposite directions.
- In fully-ionized plasmas this has been considered for plasma confinement devices.

Different approaches (fully ionized)

- Fully ionized plasma & uniform field: classical transport coefficients. Using Chapman-Enskog or Grad methods to solve Boltzman equation with Landau collisional term (Braginskii, Balescu). Fully consistent. Several calculations use these values, even in partially ionized plasmas (Spitzer conductivity).
- For non-uniform field there is no general method. In toroidal confined plasmas fully discussed: neoclassical coefficients. Include curvature and particle drifts (Pfirsch-Schlüter fluxes) and even long-range particle mean paths (banana fluxes). Average over field surfaces, no momentum equation (Balescu).
- Turbulence cannot be neglected: anomalous transport coefficients, non-thermal stationary states and turbulent energy cascades. No general formulation so far.

Different approaches (partially ionized)

- Strongly ionized plasmas: Braginskii deduction (chapter 7); with collision frequencies independent of relative velocity. Neglecting the electron inertial terms a generalized Ohm's law is obtained (thermal conductivity and thermodiffusion can not be obtained this way)
- Weakly ionized plasmas (neutrals much more abundant), only collisions with neutrals relevant, electron and ion coefficients directly from Boltzan equations (Rozhansky & Tsendin).
- General expresion for all the range of ionization not known.
- Deviations of quasi-Maxwellian distributions can be important in some cases (for example, the run-away electrons).

Conclusions

- Multi-fluid description is an step forward from the relatively simple MHD theory in describing partially ionized plasmas.
- However, for reinder it fully operational we need a set of assumptions and neglections not fully explored or understood. (collisional terms and higher order moments).
- Even the simplest way of considering the partial ionization effects (such as generalized Ohm's law and energy equation) need information about the transport coefficients.
- No general theory, different approaches might work depending on the problem

Thank you for your attention.