Numerical schemes for multifluid magnetohydrodynamics

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Giant Molecular Clouds (GMC)

e.g. Rosette Molecular Cloud

Size	$\simeq 35$ pc
Mass	$\simeq 10^5\ \mathrm{M}_\odot$
Mean Density	$\simeq 10^{-22}$ gm cm^{-3}
Temperature	$\simeq 10$ K $\Rightarrow$ sound speed $\simeq 0.2$ km s^{-1}
Alfvén speed	$\simeq 2$ km s^{-1} $\Rightarrow$ magnetic pressure dominates
Velocity dispersion	$\simeq 10$ km s^{-1}

Translucent Clumps

Rosette GMC not Homogeneous: CO maps show that it consists of $\simeq 70$ clumps with

Sizes	$\simeq 3.5$ – 8.0 pc
Masses	$\simeq 10^2$ – $2\ 10^3$ $M_\odot$
Densities	10^{-21} gm cm^{-3}
Temperature	$\simeq 10$ K $\Rightarrow$ Sound speed $\simeq 0.2$ km s^{-1}
Alfvén speed	$\simeq 2$ km s^{-1} $\Rightarrow$ magnetic pressure dominates (Crutcher 1999)
Velocity dispersion	$\simeq 1$ km s^{-1}
$\Rightarrow$ Jeans Mass	$3\ 10^3$ $M_\odot$ (based on velocity dispersion)

Dense Cores

These clumps also have substructure. Contain dense cores with

Sizes	< 1 pc
Masses	$\simeq 10 - 100$ $M_\odot$
Densities	$\simeq 10^{-19}$ gm cm^{-3}
Temperature	$\simeq 10$ K $\Rightarrow$ Sound speed $\simeq 0.2$ km s^{-1}
Alfvén speed	$\simeq 2$ km s^{-1} $\Rightarrow$ magnetic pressure dominates
Velocity dispersion	$\simeq 0.3$ km s^{-1}
$\Rightarrow$ Jeans Mass	10 $M_\odot$ (based on velocity dispersion)

Ambipolar Diffusion (Ion–Neutral Drift)

Low ionization fraction X_i $(< 10^{-4}) \rightarrow$ ambipolar diffusion.

Magnetic Reynolds No $= 1$ for

$$\text{Length scale} = 0.04 \frac{1}{M_A} \left(\frac{B}{10^{-5}}\right) \left(\frac{10^{-6}}{X_i}\right) \left(\frac{10^3}{n}\right)^{3/2} \text{pc} \quad (M_A \text{ is Alfvén Mach No} \simeq 1)$$

$\Rightarrow$ Magnetic Reynolds number < 100 in Translucent Clumps and Dense cores

$\Rightarrow$ Ambipolar Diffusion important on scales smaller than GMC.

Viscosity

In neutral gas, Reynolds No $= 1$ for

$$\text{Length scale} = 3.2\ 10^{-4} \left(\frac{1}{Mn}\right) \text{pc} \quad (M \text{ is Mach No})$$

Multifluid Equations

N fluids with equations ($i = 1 \cdots N$)

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial \rho_i v_{ix}}{\partial x} = \sum_{j \neq i} S_{ij}$$

S_{ij} – rate of conversion of i to j

$$\frac{\partial \rho_i \mathbf{v}_i}{\partial t} + \frac{\partial}{\partial x}(\rho_i v_{ix} \mathbf{v}_i + p_i \hat{\imath}) = \alpha_i \rho_i (\mathbf{E} + \mathbf{v}_i \wedge \mathbf{B}) + \sum_{j \neq 1} \mathbf{f}_{ij}$$

$\mathbf{f}_{ij}$ – force exerted on i by j, α_i – charge to mass ratio

$$\frac{\partial e_i}{\partial t} + \frac{\partial}{\partial x}[v_{ix}(\frac{1}{2}\rho_i v_i^2 + p_i)] = H_i + \sum_{j \neq i} G_{ij}$$

H_i – energy loss rate for i, G_{ij} – energy transfer rate from j to i

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \wedge \mathbf{E}, \quad \nabla \wedge \mathbf{B} = \mathbf{J} = \sum_i \alpha_i \rho_i \mathbf{v}_i$$

Species 1 - neutral ($\alpha_1 = 0$), Species $2 \cdots N$ charged.

Force is of the form

$$\mathbf{f}_{ij} = K_{ij}\rho_i\rho_j(\mathbf{v}_j - \mathbf{v}_i)$$

Define Hall parameter

$$\beta_i = \frac{\alpha_i B}{\rho_1 K_{i1}}$$

$\beta_i \gg 1 \Rightarrow$ Species i tied to field lines

$\beta_i \ll 1 \Rightarrow$ Species i tied to neutrals

in ISM $\beta \gg 1$ for ions and electrons, but not for grains

Time Dependent Numerical Scheme

Two Fluid

$\beta_i \gg 1$ for all $i > 1 \Rightarrow$ single conducting fluid.

Upwind (Godunov Type) scheme for each fluid. Add source terms. Subshocks captured in usual way.

But

Must have all Hall parameters $\beta_i \gg 1$ – true for ions and electrons, but not for grains.

If density of conducting fluid $\ll$ total density

$\Rightarrow$ conducting fluid wavespeeds $\gg$ equilibrium wavespeeds

$\Rightarrow$ small timestep with explicit scheme

Can increase mass of ions to increase timestep (Li, McKee & Klein 2006).

But only works for single conducting fluid.

Multi-Fluid

Some species with $\beta_i \simeq 1$

Total density of charged species $\ll$ total density

$\Rightarrow$ neglect inertia of charged species (otherwise equations are stiff)

$$\alpha_i \rho_i (\mathbf{E} + \mathbf{v}_i \wedge \mathbf{B}) + \sum_{j \neq 1} \mathbf{f}_{ij} = \frac{\partial \rho_i \mathbf{v}_i}{\partial t} + \frac{\partial}{\partial x}(\rho_i v_{ix} \mathbf{v}_i + p_i \hat{\imath}) \simeq 0$$

Get single fluid with induction equation

$$
\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \wedge \mathbf{E} \ &= \quad \nabla \wedge (\mathbf{v} \wedge \mathbf{B}) && \text{hyperbolic} \\
&- \quad \nabla \wedge [\nu_0 \frac{(\mathbf{J} \cdot \mathbf{B})}{B^2} \mathbf{B}] && \text{conduction parallel to field} \\
&- \quad \nabla \wedge [\nu_1 \frac{(\mathbf{J} \wedge \mathbf{B})}{B}] && \text{Hall effect} \\
&- \quad \nabla \wedge [\nu_2 \frac{(\mathbf{J} \wedge \mathbf{B})}{B^2} \wedge \mathbf{B}] && \text{ambipolar diffusion}
\end{aligned}
$$

Here $\mathbf{v}$ is neutral velocity.

Resistivities

Conductivities are

$$\sigma_0 = \frac{1}{B}\sum_i \alpha_i \rho_i \beta_i, \quad \sigma_1 = \frac{1}{B}\sum_i \frac{\alpha_i \rho_i \beta_i}{(1+\beta_i^2)}, \quad \sigma_2 = -\frac{1}{B}\sum_i \frac{\alpha_i \rho_i}{(1+\beta_i^2)}$$

Resistivities are

$$\nu_0 = \frac{1}{\sigma_0} \quad \nu_1 = -\frac{\sigma_2}{(\sigma_1^2+\sigma_2^2)} \quad \nu_2 = -\frac{\sigma_1}{(\sigma_1^2+\sigma_2^2)}$$

Note $|\nu_1| \ll 1$ if all $\beta_i \gg 1$ i.e. no Hall effect

To compute these need charged species densities, ρ_i.

Momentum equations for charged species reduce to

$$\frac{\beta_i}{B}(\mathbf{E} + \mathbf{v}_i \wedge \mathbf{B}) + (\mathbf{v}_1 - \mathbf{v}_i) = 0 \quad i = 2 \cdots N$$

(Neglecting inertia and collisions between charged species)

Also have

$$\mathbf{J} = \nabla \wedge \mathbf{B} = \sum_i \alpha_i \rho_i \mathbf{v}_i$$

These N equations determine $\mathbf{E}$ and the $\mathbf{v}_i$ for $i = 2 \cdots N$.

Given the $\mathbf{v}_i$, determine the ρ_i from the continuity equations

Subtleties

If not isothermal, must include Lorentz force, $\mathbf{J} \wedge \mathbf{B}$ as source term in momentum and energy equations to get correct relations across subshock.

Hall term dispersive with

$$\omega^2 = \nu_1^2 \cos^2 \theta k^4 \quad (\theta \text{ is angle between field and x axis})$$

i.e. phase and group velocity $\to \infty$ as wavelength $\to 0$ (whistler waves).

Might suppose that group velocity, $2\nu_1 \cos \theta k$, is effective wavespeed and Δx is smallest wavelength

$\Rightarrow$ stable timestep for explicit scheme $\Delta t = \dfrac{\Delta x^2}{4\pi\nu_1 \cos\theta}$.

But

Obvious explicit scheme unconditionally unstable for pure Hall effect $\Rightarrow$

either implicit scheme for resistive terms

or differencing in O'Sullivan & Downes 2006 and super-time-stepping

Algorithm

1) Calculate solution at half time using a first order scheme which is explicit for hyperbolic terms, implicit for resistive terms.
2) Use this to calculate explicit, second order accurate fluxes for both hyperbolic and resistive terms.
3) Advance solution by complete timestep using these fluxes.

$\Rightarrow$

scheme is second order and stability limited by hyperbolic timestep, not resistive timestep, even if Hall term is dominant.

Shock Structure with Large Hall Parameters

Two charged species:

$$\beta_2 = -5.8\ 10^6 \text{ (electrons)}, \beta_3 = 5.8\ 10^3 \text{ (ions)}$$

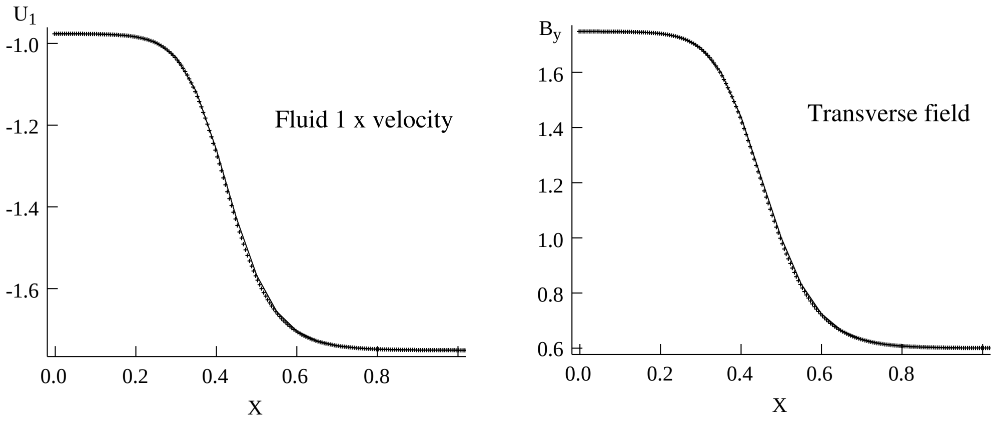
Preshock state:

$B_x = 1.0$, $B_y = 0.6$, Fast shock with Fast Mach No = 1.5

$\nu_0 = 1.7\ 10^{-12}$, $\nu_1 = 10^{-5}$, $\nu_2 = -0.058$ (Hall effect negligible)

Isothermal – neutral pressure negligible.

High Resolution

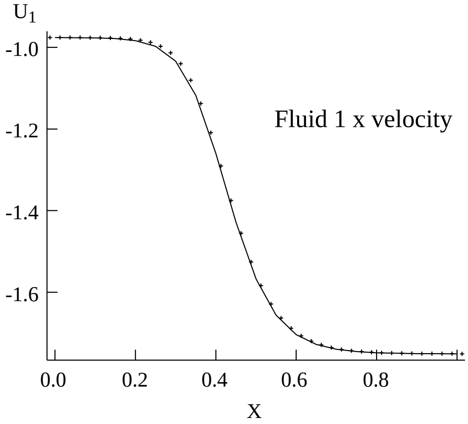


$\Delta x = 5\ 10^{-3}$.

Line – Integration of steady equations, markers – Numerical scheme

No rotation – Z component of field $\simeq 10^{-4}$

Low Resolution



$\Delta x = 2.5\ 10^{-2}.$

Line – Integration of steady equations, markers – Numerical scheme

Shock Structure with Strong Hall Effect

Two charged species:

$$\beta_2 = -5.8\ 10^6 \text{ (electrons)},\ \beta_3 = 0.233 \text{ (grains)}.$$

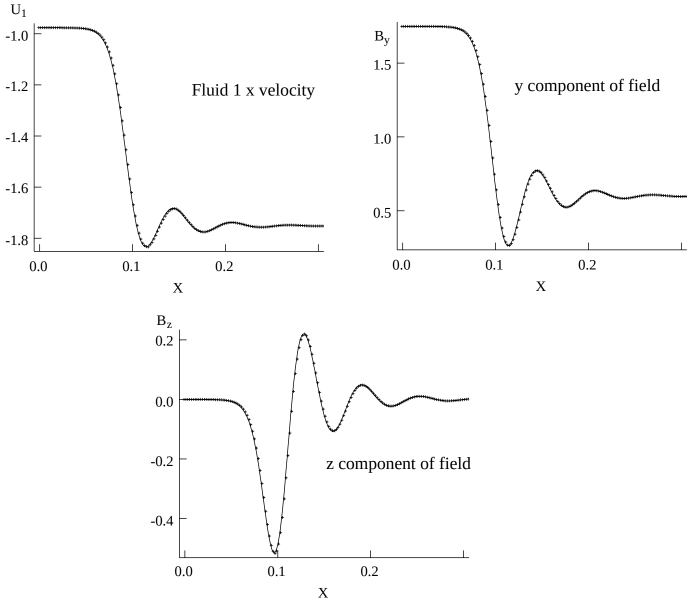
Preshock:

$B_x = 1.0$, $B_y = 0.6$, Fast shock with Fast Mach No = 1.5

Preshock $\nu_0 = 1.7\ 10^{-9}$, $\nu_1 = 0.01$, $\nu_2 = 0.0023$ (Significant Hall effect)

Isothermal – neutral pressure negligible.

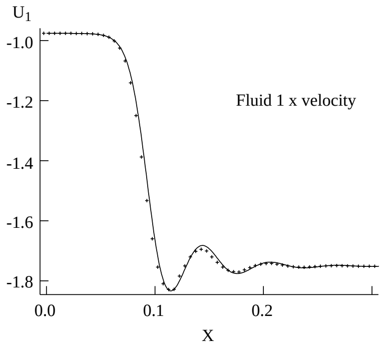
High Resolution



$\Delta x = 2\ 10^{-3}$.

Line – Integration of steady equations, markers – Numerical scheme

Low Resolution



$$\Delta x = 5\ 10^{-3}$$

Line – Integration of steady equations, markers – Numerical scheme

Shock Structure with Neutral Subshock

Two charged species:

$$\beta_2 = -5.8\ 10^6 \text{ (electrons)}, \ \beta_3 = 5.8\ 10^3 \text{ (ions)}$$

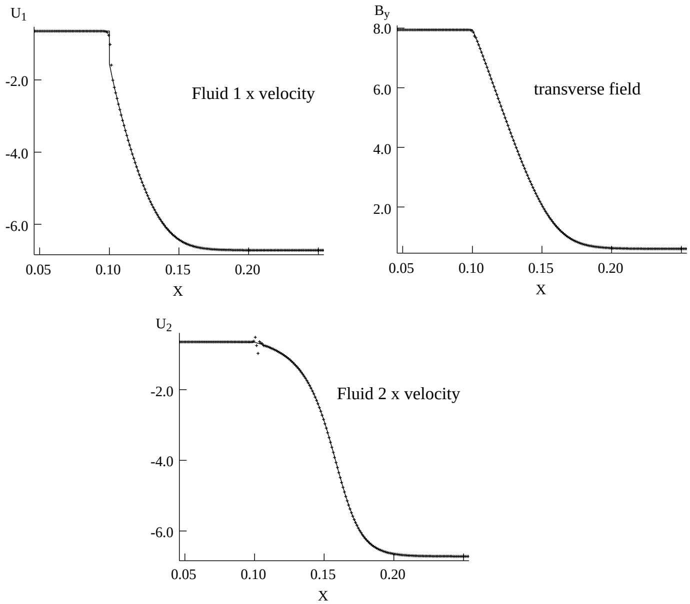
Preshock state:

$B_x = 1.0$, $B_y = 0.6$, Fast shock with Fast Mach No = 5

$\nu_0 = 1.7\ 10^{-12}$, $\nu_1 = 10^{-5}$, $\nu_2 = -0.058$ (Hall effect negligible)

Isothermal – neutral sound speed $a = 1$.

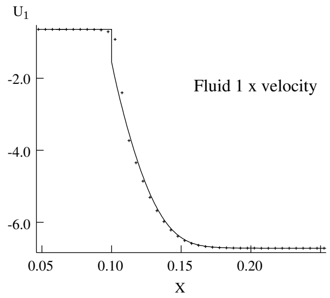
High Resolution



$\Delta x = 10^{-3}$.

Line – Integration of steady equations, markers – Numerical scheme

Low Resolution



$\Delta x = 5\ 10^{-3}$.

Line – Integration of steady equations, markers – Numerical scheme

Multidimensions

Resistive terms contain cross-derivatives

$\Rightarrow$

fully implicit scheme messy.

But

Can treat cross-derivatives explicitly and only use implicit approximation for diagonal terms:

$$\frac{\partial^2 B_y}{\partial x^2}, \quad \frac{\partial^2 B_x}{\partial y^2} \quad \text{etc}$$

Scheme then has same stability properties as in one dimension. Cheap because just have tridiagonal matrices to invert.

Can use scheme for:

1. Stability of multifluid shocks (Wardle instability)
2. Ambipolar diffusion in star forming regions.
3. Ambipolar diffusion and Hall effect in accretion discs
4. etc