Two-fluid MHD approach for partially ionized solar plasmas

T.V. Zaqarashvili and M.L. Khodachenko

Space Research Institute of Austrian Academy of Sciences,
Graz, Austria
Solar atmospheric model

FAL93-3 model (Fontenla et al. 1993)
Solar atmospheric model

FAL93-3 model (Fontenla et al. 1993)

- $n_e$ – electron number density
- $n_H$ – neutral hydrogen number density
- $n_{\text{He}}$ – neutral helium number density

Plasma is only weakly ionized in the photosphere, but becomes almost fully ionized in the transition region and corona.
Neutral atoms may change the plasma dynamics through collisions with charged particles:


Influence on energy flux of Alfvén waves in the photosphere (Vranjes et al. 2008);


Emerging magnetic flux tubes (Leake & Arber 2005, Arber et al. 2007);

Influence on resonant absorption (Soler et al. 2009).

Influence on Kelvin-Helmholtz instability (Soler et al. 2012).
Three-Fluid equations

\[
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{V}_e) = 0,
\]

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{V}_i) = 0,
\]

\[
\frac{\partial n_n}{\partial t} + \nabla \cdot (n_n \mathbf{V}_n) = 0,
\]

\[
m_e n_e \left( \frac{\partial \mathbf{V}_e}{\partial t} + (\mathbf{V}_e \cdot \nabla) \mathbf{V}_e \right) = -\nabla p_e - \nabla \cdot \pi_e - e n_e \left( \mathbf{E} + \frac{1}{c} \mathbf{V}_e \times \mathbf{B} \right) + \mathbf{R}_e,
\]

\[
m_i n_i \left( \frac{\partial \mathbf{V}_i}{\partial t} + (\mathbf{V}_i \cdot \nabla) \mathbf{V}_i \right) = -\nabla p_i - \nabla \cdot \pi_i - e n_i \left( \mathbf{E} + \frac{1}{c} \mathbf{V}_i \times \mathbf{B} \right) + \mathbf{R}_i,
\]

\[
m_n n_n \left( \frac{\partial \mathbf{V}_n}{\partial t} + (\mathbf{V}_n \cdot \nabla) \mathbf{V}_n \right) = -\nabla p_n - \nabla \cdot \pi_n + \mathbf{R}_n,
\]

\[
\frac{3}{2} n_e k \left( \frac{\partial T_e}{\partial t} + (\mathbf{V}_e \cdot \nabla) T_e \right) + p_e \nabla \cdot \mathbf{V}_e + \pi_e : \nabla \mathbf{V}_e = -\nabla \cdot \mathbf{q}_e + Q_e,
\]

\[
\frac{3}{2} n_i k \left( \frac{\partial T_i}{\partial t} + (\mathbf{V}_i \cdot \nabla) T_i \right) + p_i \nabla \cdot \mathbf{V}_i + \pi_i : \nabla \mathbf{V}_i = -\nabla \cdot \mathbf{q}_i + Q_i,
\]

\[
\frac{3}{2} n_n k \left( \frac{\partial T_n}{\partial t} + (\mathbf{V}_n \cdot \nabla) T_n \right) + p_n \nabla \cdot \mathbf{V}_n + \pi_n : \nabla \mathbf{V}_n = -\nabla \cdot \mathbf{q}_n + Q_n,
\]

\[
p_e = n_e k T_e, \quad p_i = n_i k T_i, \quad p_n = n_n k T_n,
\]

\[
\mathbf{R}_a \quad \text{is the change of impulse,} \quad \mathbf{q}_a \quad \text{is the heat flux density,} \quad Q_a \quad \text{is the heat production.}
\]
The description of the system is completed by Maxwell equations

\[ \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \]

\[ \nabla \times \vec{B} = -\frac{4\pi}{c} \vec{j}, \]

where

\[ \vec{j} = -en_e (\vec{V}_e - \vec{V}_i) \]

is the current density and

\[ \nabla \cdot \vec{B} = 0. \]

Plasma is supposed to be quasi neutral

\[ n_e = n_i. \]
Impulse change and heat production can be expressed as (Braginskii 1965)

\[
\vec{R}_e = -\alpha_{ei} (\vec{V}_e - \vec{V}_i) - \alpha_{en} (\vec{V}_e - \vec{V}_n),
\]

\[
\vec{R}_i = -\alpha_{ie} (\vec{V}_i - \vec{V}_e) - \alpha_{in} (\vec{V}_i - \vec{V}_n),
\]

\[
\vec{R}_n = -\alpha_{ne} (\vec{V}_n - \vec{V}_e) - \alpha_{ni} (\vec{V}_n - \vec{V}_i),
\]

\[
Q_e = \alpha_{ei} (\vec{V}_e - \vec{V}_i) \vec{V}_e + \alpha_{en} (\vec{V}_e - \vec{V}_n) \vec{V}_e,
\]

\[
Q_i = \alpha_{ie} (\vec{V}_i - \vec{V}_e) \vec{V}_i + \alpha_{in} (\vec{V}_i - \vec{V}_n) \vec{V}_i,
\]

\[
Q_n = \alpha_{ne} (\vec{V}_n - \vec{V}_e) \vec{V}_n + \alpha_{ni} (\vec{V}_n - \vec{V}_i) \vec{V}_n,
\]

\[\alpha_{ab} = \alpha_{ba}\] are coefficients of friction between different sort of particles.

For time scales longer than ion-electron collision time, the ion-electron gas can be considered as a single fluid.

Then, any additional sort of neutral atoms can be treated as a separate fluid.
The coefficient of friction between ions and electrons can be expressed as (Braginskii 1965)

\[ \alpha_{ie} = \frac{4\sqrt{2\pi}\lambda e^4 n_i n_e m_{ie}}{3\sqrt{m_{ie}} (kT_e)^{3/2}}, \]

where \( \lambda \) is the Coulomb logarithm.

The coefficient of friction between ions and neutral hydrogen atoms is (Braginskii 1965)

\[ \alpha_{in} = n_i n_n m_{in} \sigma_{in} \sqrt{\frac{8kT}{\pi m_{in}}}, \]

where \( \sigma_{in} \) is ion-hydrogen collision cross-section and \( m_{in} \) is the reduced mass.

For elastic collision, the ion-hydrogen collision cross-section is (Braginskii 1965)

\[ \sigma_{in} = 2\pi (r_i + r_n)^2, \]

which approximately equals atomic cross section.
Ion-electron collision frequency is expressed by

\[ \nu_{ie} = \frac{\alpha_{ie}}{m_e n_e} = \frac{4 \sqrt{2 \pi \lambda} e^4 n_i}{3 \sqrt{m_e (kT_e)^{3/2}}}. \]

Ion-neutral collision frequency is often expressed as

\[ \nu_{in} = \frac{\alpha_{in}}{m_i n_i}, \]

while neutral-ion collision frequency is often expressed as

\[ \nu_{ni} = \frac{\alpha_{in}}{m_n n_n}. \]

According to this formulation, ion-neutral collision frequency is different than neutral-ion collision frequency. But, the mean collision frequency between ions and neutral atoms should be a single value due to physical basis.

From simple equations of motion of ions and neutals one can derive the equation for relative velocity between ions and neutals

\[ \frac{\partial (\vec{V}_i - \vec{V}_n)}{\partial t} = -\alpha_{in} \left( \frac{1}{m_i n_i} + \frac{1}{m_n n_n} \right) (\vec{V}_i - \vec{V}_n). \]

This equation gives a single value for the ion-neutral collision frequency (Zaqarashvili et al. 2011) \[ \nu_{in} = \nu_{ni} = \alpha_{in} \left( \frac{1}{m_i n_i} + \frac{1}{m_n n_n} \right) = 2(n_i + n_n) \sigma_{in} \sqrt{\frac{kT}{\pi m_i}}. \]
Let us use FAL93-3 model (Fontenla et al. 1993) for temperature and number densities:

<table>
<thead>
<tr>
<th></th>
<th>z=0</th>
<th>z=900</th>
<th>z=1900</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>6520 K</td>
<td>6000 K</td>
<td>8900 K</td>
</tr>
<tr>
<td>$n_e$</td>
<td>$7.67 \times 10^{13}$ cm$^{-3}$</td>
<td>$2.60 \times 10^{11}$ cm$^{-3}$</td>
<td>$1.27 \times 10^{11}$ cm$^{-3}$</td>
</tr>
<tr>
<td>$n_i$</td>
<td>$5.99 \times 10^{13}$ cm$^{-3}$</td>
<td>$2.43 \times 10^{11}$ cm$^{-3}$</td>
<td>$1.20 \times 10^{11}$ cm$^{-3}$</td>
</tr>
<tr>
<td>$n_n$</td>
<td>$1.18 \times 10^{17}$ cm$^{-3}$</td>
<td>$8.95 \times 10^{13}$ cm$^{-3}$</td>
<td>$1.79 \times 10^{11}$ cm$^{-3}$</td>
</tr>
</tbody>
</table>

$\nu_{ie}$ = $8 \times 10^8$ Hz \hspace{1cm} 10$^6$ Hz \hspace{1cm} 3.7 $10^6$ Hz

$\nu_{in}$ = $8.6 \times 10^6$ Hz \hspace{1cm} 6.2 $10^3$ Hz \hspace{1cm} 24 Hz

Ion-electron collision frequency seems to be always much higher than ion-neutral collision frequency in the whole atmosphere.

However, this statement may be overestimated in the lower photosphere, where significant number of heavy ions exists.
Two-Fluid MHD equations

\[ \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{V}_i) = 0, \]

\[ \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{V}_e) = 0, \]

\[ m_i n_i \left( \frac{\partial \vec{V}_i}{\partial t} + (\vec{V}_i \cdot \nabla) \vec{V}_i \right) = -\nabla p_{ie} - \frac{1}{c} \vec{j} \times \vec{B} + \frac{\alpha_{en}}{e n_e} \vec{j} - (\alpha_{in} + \alpha_{en}) (\vec{V}_i - \vec{V}_n), \]

\[ m_e n_e \left( \frac{\partial \vec{V}_e}{\partial t} + (\vec{V}_e \cdot \nabla) \vec{V}_e \right) = -\nabla p_e - \frac{\alpha_{en}}{e n_e} \vec{j} + (\alpha_{in} + \alpha_{en}) (\vec{V}_i - \vec{V}_n), \]

\[ \frac{\partial p_{ie}}{\partial t} + (\vec{V}_i \cdot \nabla) p_{ie} + \gamma p_{ie} \nabla \cdot \vec{V}_i = (\gamma - 1) \frac{\alpha_{ei}}{e^2 n_e^2} \vec{j}^2 + (\gamma - 1) \alpha_{in} (\vec{V}_i - \vec{V}_n) \cdot \vec{V}_i + \]

\[ + (\gamma - 1) \alpha_{en} (\vec{V}_e - \vec{V}_n) \cdot \vec{V}_e + \frac{(\vec{j} \cdot \nabla) p_e}{e n_e} + \gamma p_e \nabla \cdot \vec{j} - (\gamma - 1) \nabla \cdot (\vec{q}_i + \vec{q}_e), \]

\[ \frac{\partial p_n}{\partial t} + (\vec{V}_n \cdot \nabla) p_n + \gamma p_n \nabla \cdot \vec{V}_n = -(\gamma - 1) \alpha_{in} (\vec{V}_i - \vec{V}_n) \cdot \vec{V}_n - (\gamma - 1) \alpha_{en} (\vec{V}_e - \vec{V}_n) \cdot \vec{V}_n - (\gamma - 1) \nabla \cdot \vec{q}_n, \]

\[ p_{ie} = p_i + p_e \] is the pressure of electron-ion gas.

(Zaqareshvili et al. 2011)
Ohm’s law is obtained from the electron equation after neglecting the electron inertia

\[
\vec{E} + \frac{1}{c} \vec{V}_i \times \vec{B} + \frac{1}{en_e} \nabla p_e = \frac{\alpha_{ei} + \alpha_{en}}{e^2 n_e^2} (\vec{V}_i - \vec{V}_n) + \frac{1}{cen_e} \vec{j} \times \vec{B}.
\]

Faraday’s law and Ohm’s law lead to the induction equation

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V}_i \times \vec{B}) + \nabla \times \left( \frac{c \nabla p_e}{en_e} \right) - \nabla \times (\eta \nabla \times \vec{B}) - \nabla \times \left( \frac{\vec{j} \times \vec{B}}{en_e} \right) + \nabla \times \left( \frac{c \alpha_{en} (\vec{V}_i - \vec{V}_n)}{en_e} \right),
\]

where

\[
\eta = \frac{c^2}{4\pi \sigma} = \frac{c^2 (\alpha_{ei} + \alpha_{en})}{4\pi e^2 n_e^2}
\]

is the coefficient of magnetic diffusion.
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0,
\]
\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = -\nabla p + \frac{1}{c} \mathbf{j} \times \mathbf{B} - \nabla \cdot (\xi_i \xi_n \rho \mathbf{w} \mathbf{w}),
\]
\[
\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w} \cdot \nabla) \mathbf{V} + (\mathbf{V} \cdot \nabla) \mathbf{w} + \xi_n (\mathbf{X} \cdot \nabla) \mathbf{w} - (\mathbf{w} \cdot \nabla) \zeta_i \mathbf{w} = -\left( \frac{\nabla p_{ie}}{\rho \xi_i} - \frac{\nabla p_{en}}{\rho \xi_n} \right) + \frac{1}{c \rho \xi_i} \mathbf{j} \times \mathbf{B} + \frac{\alpha_{en}}{\rho \xi_i \xi_n} \mathbf{j} - \frac{\alpha_{in} + \alpha_{en}}{\rho \xi_i \xi_n} \mathbf{w},
\]
\[
\frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla) p + \gamma \rho \nabla \cdot \mathbf{V} - \xi_i (\mathbf{w} \cdot \nabla) p - \gamma \rho \nabla \cdot (\xi_i \mathbf{w}) + (\mathbf{w} \cdot \nabla) p_{ie} + \gamma \rho_{ie} \nabla \cdot \mathbf{w} = \left( \gamma - 1 \right) \frac{\alpha_{ie} + \alpha_{en}}{\epsilon^2 n_e^2} \mathbf{j}^2 + \left( \gamma - 1 \right) \left( \alpha_{in} + \alpha_{en} \right) \mathbf{w}^2 - \left( \gamma - 1 \right) \frac{2 \alpha_{en}}{\epsilon n_e} \mathbf{j} \cdot \mathbf{w} + \left( \frac{\mathbf{j} \cdot \nabla}{\epsilon n_e} \right) + \gamma \rho \nabla \cdot \frac{\mathbf{j}}{\epsilon n_e} - \left( \gamma - 1 \right) \nabla \cdot (\mathbf{q}_i + \mathbf{q}_e + \mathbf{q}_n),
\]
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \nabla \times \left( \frac{c \nabla p_{ie}}{\epsilon n_e} \right) - \nabla \times (\eta \nabla \times \mathbf{B}) - \nabla \times \left( \frac{\mathbf{j} \times \mathbf{B}}{\epsilon n_e} \right) + \nabla \times \left( \frac{c \alpha_{en} \mathbf{w}}{\epsilon n_e} \right) + \nabla \times (\xi_n \mathbf{w} \times \mathbf{B}),
\]
where
\[
\mathbf{V} = \frac{\rho_i \mathbf{V}_i + \rho_n \mathbf{V}_n}{\rho_i + \rho_n}
\]
is the total velocity, \( \mathbf{w} = \mathbf{V}_i - \mathbf{V}_n \) is the relative velocity.

For time scales longer than ion-neutral collision time inertial terms can be neglected.

\[
\mathbf{w} = -\frac{\mathbf{\tilde{G}}}{\alpha_{in} + \alpha_{en}} - \left( \frac{\nabla p_{ie}}{\rho \xi_i} - \frac{\nabla p_{en}}{\rho \xi_n} \right) + \frac{\xi_n}{c(\alpha_{in} + \alpha_{en})} \mathbf{j} \times \mathbf{B} + \frac{\alpha_{en}}{\epsilon n_{en} (\alpha_{in} + \alpha_{en})} \mathbf{j}.
\]
Then we obtain

\[
\vec{w} = -\frac{\vec{G}}{\alpha_{\text{in}} + \alpha_{\text{en}}} - \left(\frac{\nabla p_{\text{ie}} - \nabla p_n}{\rho \xi_i \rho \xi_n}\right) + \frac{\xi_n}{c(\alpha_{\text{in}} + \alpha_{\text{en}})} \vec{j} \times \vec{B} + \frac{\alpha_{\text{en}}}{\epsilon_{\text{en}}(\alpha_{\text{in}} + \alpha_{\text{en}})} \vec{j},
\]

and the induction equation becomes

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{V} \times \vec{B}) + \frac{c}{e} \nabla \times \left(\frac{\rho \nabla p_e - \varepsilon \vec{G}}{n_e}\right) - \nabla \times (\eta_{\text{ie}} \nabla \times \vec{B}) - \frac{c}{4\pi e} \nabla \times \left(\frac{1 - 2\varepsilon n_e}{n_e} (\nabla \times \vec{B}) \times \vec{B}\right) - \\
- \nabla \times \left(\frac{\varepsilon n_e}{\alpha_{\text{in}} + \alpha_{\text{en}}} \vec{G} \times \vec{B}\right) + \nabla \times \left(\frac{\xi_n^2}{4\pi(\alpha_{\text{in}} + \alpha_{\text{en}})} ((\nabla \times \vec{B}) \times \vec{B}) \times \vec{B}\right),
\]

where

\[
\eta_{\text{ie}} = \frac{c^2}{4\pi e^2 n_e^2} \left(\alpha_{\text{ei}} + \alpha_{\text{en}} - \frac{\alpha_{\text{en}}^2}{\alpha_{\text{in}} + \alpha_{\text{en}}}\right), \quad \vec{G} = \xi_n \nabla p_{\text{ie}} - \xi_i \nabla p_n \quad \varepsilon = \frac{\alpha_{\text{en}}}{\alpha_{\text{in}} + \alpha_{\text{en}}}.\]

Next we consider MHD waves in the two-fluid approach.
We consider linear Alfvén waves in homogeneous medium with uniform magnetic field

\[
\frac{\partial u_{iy}}{\partial t} = \frac{B_z}{4\pi \rho_i} \frac{\partial b_y}{\partial z} - \frac{\alpha_{in}}{\rho_i} (u_{iy} - u_{ny}),
\]

\[
\frac{\partial u_{ny}}{\partial t} = \frac{\alpha_{in}}{\rho_n} (u_{iy} - u_{ny}),
\]

\[
\frac{\partial b_y}{\partial t} = B_z \frac{\partial u_{iy}}{\partial z}.
\]

Fourier analyses with \( \exp(ik_z z - i\omega t) \) gives the dispersion relation

\[
\omega^3 + i \frac{\alpha_{in} \rho_0}{\rho_i \rho_n} \omega^2 - k_z^2 V_A^2 \frac{\rho_0}{\rho_i} \omega - ik_z^2 V_A^2 \frac{\alpha_{in} \rho_0}{\rho_i \rho_n} = 0.
\]

The dispersion relation has three solutions: two Alfvén waves damped by ion-neutral collision and purely imaginary solution, which is the vortex mode connected with the vorticity of neutral fluid.

The vortex mode has zero frequency in ideal fluid, but gains imaginary part when collisions between ions and neutrals are considered.
Alfvén waves in two-fluid approach

Frequency and damping rate are normalized by $k_z V_A$ and $a = \frac{k_z V_A \rho_0}{\alpha_{in}}$ is normalized Alfvén frequency.

Zaqarashvili et al. 2011
Linear magneto-acoustic waves in two-fluid approach

\[ \frac{\partial \tilde{\rho}_i}{\partial t} + \rho_i \left( \frac{\partial V_{ix}}{\partial x} + \frac{\partial V_{iz}}{\partial z} \right) = 0, \]

\[ \frac{\partial \tilde{\rho}_n}{\partial t} + \rho_n \left( \frac{\partial V_{nx}}{\partial x} + \frac{\partial V_{nz}}{\partial z} \right) = 0, \]

\[ \frac{\partial V_{ix}}{\partial t} = - \frac{c_{si}^2}{\rho_i} \frac{\partial \tilde{\rho}_i}{\partial x} - \frac{B_z}{4\pi \rho_i} \frac{\partial b_{ix}}{\partial x} + \frac{B_z}{4\pi \rho_i} \frac{\partial b_x}{\partial z} - \frac{\alpha_{in}}{\rho_i} (V_{ix} - V_{nx}), \]

\[ \frac{\partial V_{iz}}{\partial t} = - \frac{c_{si}^2}{\rho_i} \frac{\partial \tilde{\rho}_i}{\partial z} - \frac{\alpha_{in}}{\rho_i} (V_{iz} - V_{nz}), \]

\[ \frac{\partial V_{nx}}{\partial t} = - \frac{c_{sn}^2}{\rho_n} \frac{\partial \tilde{\rho}_n}{\partial x} + \frac{\alpha_{in}}{\rho_n} (V_{ix} - V_{nx}), \]

\[ \frac{\partial V_{nz}}{\partial t} = - \frac{c_{sn}^2}{\rho_n} \frac{\partial \tilde{\rho}_n}{\partial z} + \frac{\alpha_{in}}{\rho_n} (V_{iz} - V_{nz}), \]

\[ \frac{\partial b_x}{\partial t} = B_z \frac{\partial u_{ix}}{\partial z}. \]

After Fourier analyses we obtain the dispersion relation for fast and slow MHD waves.

Fast waves are similar to Alfvén waves, while slow waves have different properties.
Damping rate of slow waves is different when it is derived from the energy equation (Braginskii 1965) and through a normal mode analysis (Forteza et al. 2007).

Slow magneto-acoustic waves show damping for purely parallel propagation in the case of Braginskii, while the damping is absent in Forteza et al. (2007).
Red asterisks corresponds to two-fluid solution, dashed line corresponds to Braginskii (1965). The damping rate is zero in Forteza et al. (2007).
Conclusions

- Two-fluid MHD approximation, where electron+ion gas and neutral atoms are the two components, is important when time scales approach to ion-neutral collision time;
- Single-fluid approach is a good approximation for longer times scales;
- Damping rates of Alfvén and fast magneto-acoustic waves reach their maximum near ion-neutral collision frequency, but decrease for higher frequency harmonics of the wave spectrum;
- No cut-off wave number of MHD waves appears in the two-fluid approach;
- The damping of slow magneto-acoustic waves is not zero in the parallel propagation in two-fluid approach in coincidence with Braginskii (1965);
- The two-fluid approach could be important for instability processes, when unstable harmonics have short spatial scales.
Thank you for attention!